

# AUTOMORPHISMS OF THE TWO-PARAMETER HOPF ALGEBRA $\check{U}_{r,s}^{\geq 0}(G_2)$

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**ABSTRACT.** We determine the group of algebra automorphisms for the two-parameter quantized enveloping algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . As an application, we prove that the group of Hopf algebra automorphisms for  $\check{U}_{r,s}^{\geq 0}(G_2)$  is isomorphic to a torus of rank two.

## INTRODUCTION

Let  $\mathfrak{g}$  be a finite dimensional complex simple Lie algebra and let  $r, s \in \mathbb{C}^*$ . The two-parameter quantized enveloping algebras (or quantum groups)  $U_{r,s}(\mathfrak{g})$  have been studied in the literatures [2, 3, 1] and the references therein. Recently, more studies have been conducted toward their subalgebras such as  $U_{r,s}^+(\mathfrak{g})$ , and the augmented Hopf algebras  $\check{U}_{r,s}^{\geq 0}(\mathfrak{g})$ . In [7], the author has computed the derivations for the subalgebra  $U_{r,s}^+(sl_3)$ , and determined both the algebra automorphism group and Hopf algebra automorphism group for the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(sl_3)$ . A similar work has been carried out for the algebra  $U_{r,s}^+(B_2)$  and the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(B_2)$  in [8]. The results in these works suggest that the subalgebras  $U_{r,s}^+(\mathfrak{g})$  and  $\check{U}_{r,s}^{\geq 0}(\mathfrak{g})$  are close analogues of their one-parameter counterparts, which facilitates further investigation toward these subalgebras.

In this paper, we are planning to derive some similar results for the two-parameter Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$  in terms of its (Hopf) algebra automorphisms. In particular, we will first determine the group of algebra automorphisms for the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . Then, as an application, we will further prove that the group of Hopf algebra automorphisms for  $\check{U}_{r,s}^{\geq 0}(G_2)$  is indeed isomorphic to a torus of rank 2. We will closely follow the approach used in [4].

The paper is organized as follows. In Section 1, we will recall some basics on the two-parameter Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . In Section 2, we

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will determine the group of algebra automorphisms for the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . In Section 3, we will determine the group of Hopf algebra automorphisms for the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ .

### 1. SOME BASIC PROPERTIES OF THE HOPF ALGEBRA $\check{U}_{r,s}^{\geq 0}(G_2)$

Recall that the two-parameter quantum group  $U_{r,s}(G_2)$  associated to the finite dimensional complex simple Lie algebra of type  $G_2$  has been studied in [5, 6]. In particular, a PBW basis of  $U_{r,s}(G_2)$  has been constructed in [6]. For the readers' convenience, we will recall the construction of the subalgebra  $U_{r,s}^+(G_2)$  together with some of its basic properties from [5]. In the rest of this paper, we will always assume that the parameters  $r, s$  are chosen from  $\mathbb{C}^*$  such that  $r^m s^n = 1$  implies  $m = n = 0$ .

First of all, we need to recall the following definition from the references [5, 6]:

**Definition 1.1.** The two-parameter quantized enveloping algebra  $U_{r,s}^+(G_2)$  is defined to be the  $\mathbb{C}$ -algebra generated by the generators  $e_1, e_2$  subject to the following relations:

$$\begin{aligned} e_2^2 e_1 - (r^{-3} + s^{-3}) e_2 e_1 e_2 + r^{-3} s^{-3} e_1 e_2^2 &= 0, \\ e_1^4 e_2 - (r + s)(r^2 + s^2) e_1^3 e_2 e_1 + r s(r^2 + r s + s^2) e_1^2 e_2 e_1^2 \\ - r^3 s^3 (r + s)(r^2 + s^2) e_1 e_2 e_2^3 + r^6 s^6 e_2 e_1^4 &= 0. \end{aligned}$$

In the rest of this section, we will establish some basic properties of the algebra  $U_{r,s}^+(G_2)$  and introduce an augmented Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . In particular, we will recall the construction of a PBW basis for the algebra  $U$ .

In order to recall the construction of a PBW basis of  $U_{r,s}^+(G_2)$ , we need to fix some variables. We will follow the notation in [6].

$$\begin{aligned} X_6 &= E_1 = e_1, & X_1 &= E_2 = e_2, \\ X_2 &= E_{12} = e_1 e_2 - s^3 e_2 e_1, \\ X_4 &= E_{112} = e_1 E_{12} - r s^2 E_{12} e_1, \\ X_5 &= E_{1112} = e_1 E_{112} - r^2 s E_{112} e_1, \\ X_3 &= E_{11212} = E_{112} E_{12} - r^2 s E_{12} E_{112}. \end{aligned}$$

Now we can have the following result

**Theorem 1.1.** (**Theorem 2.4.** in [6]) *The following set*

$$\{E_2^{n_1} E_{12}^{n_2} E_{11212}^{n_3} E_{112}^{n_4} E_{1112}^{n_5} E_1^{n_6} \mid n_i \in \mathbb{Z}_{\geq 0}\}$$

*forms a Lyndon basis of the algebra  $U_{r,s}^+(G_2)$ .*

□

We now recall the definition of the Hopf subalgebra  $U_{r,s}^{\geq 0}(G_2)$  from [5, 6]. We shall have the following definition.

**Definition 1.2.** The Hopf algebra  $U_{r,s}^{\geq 0}(G_2)$  is defined to be the  $\mathbb{C}$ –algebra generated by the generators  $e_1, e_2$  and  $w_1, w_2$  subject to the following relations:

$$\begin{aligned} w_1 w_1^{-1} &= w_2 w_2^{-1} = 1, & w_1 w_2 &= w_2 w_1; \\ w_1 e_1 &= r s^{-1} e_1 w_1, & w_1 e_2 &= s^3 e_2 w_1; \\ w_2 e_1 &= r^{-3} e_1 w_2, & w_2 e_2 &= r^3 s^{-3} e_2 w_2; \\ e_2^2 e_1 - (r^{-3} + s^{-3}) e_2 e_1 e_2 + r^{-3} s^{-3} e_1 e_2^2 &= 0; \\ e_1^4 e_2 - (r + s)(r^2 + s^2) e_1^3 e_2 e_1 + r s(r^2 + r s + s^2) e_1^2 e_2 e_1^2 \\ &\quad - r^3 s^3 (r + s)(r^2 + s^2) e_1 e_2 e_2^3 + r^6 s^6 e_2 e_1^4 = 0. \end{aligned}$$

Let us set

$$\begin{aligned} \Delta(e_1) &= e_1 \otimes 1 + w_1 \otimes e_1; \\ \Delta(e_2) &= e_2 \otimes 1 + w_2 \otimes e_2; \\ \Delta(w_1) &= w_1 \otimes w_1, & \Delta(w_2) &= w_2 \otimes w_2; \\ S(e_1) &= -w_1 e_1, & S(e_2) &= -w_2 e_2; \\ S(w_1) &= w_1^{-1}, & S(w_2) &= w_2^{-1}; \\ \epsilon(e_1) &= \epsilon(e_2) = 0, & \epsilon(w_1) &= \epsilon(w_2) = 1. \end{aligned}$$

Then, it is easy to see that the above operators define a Hopf algebra structure on the  $U_{r,s}^{\geq 0}(G_2)$ ; and we further have the following proposition:

**Proposition 1.1.** *The set*

$$\{X_1^a X_2^b X_3^c X_4^d X_5^e X_6^f w_1^m w_2^n | a, b, c, d, e, f \in \mathbb{Z}_{\geq 0}, m, n \in \mathbb{Z}\}$$

*forms a PBW-basis of the Hopf algebra  $U_{r,s}^{\geq 0}(G_2)$ .*

□

To define the augmented Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ , let us set the following new variables

$$k_1 = w_1^2 w_2, \quad k_2 = w_1^3 w_2^2.$$

Now we have the following definition of the augmented Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ .

**Definition 1.3.** The Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$  is defined to be the  $\mathbb{C}$ –algebra generated by the generators  $e_1, e_2$  and  $k_1, k_2$  subject to the following

relations:

$$\begin{aligned}
k_1 k_1^{-1} &= k_2 k_2^{-1} = 1, & k_1 k_2 &= k_2 k_1; \\
k_1 e_1 &= r^{-1} s^{-2} e_1 k_1, & k_1 e_2 &= r^3 s^3 e_2 k_1; \\
k_2 e_1 &= r^{-3} s^{-3} e_1 k_2, & k_2 e_2 &= r^6 s^3 e_2 k_2; \\
e_2^2 e_1 - (r^{-3} + s^{-3}) e_2 e_1 e_2 + r^{-32} s^{-3} e_1 e_2^2 &= 0; \\
e_1^4 e_2 - (r + s)(r^2 + s^2) e_1^3 e_2 e_1 + r s(r^2 + r s + s^2) e_1^2 e_2 e_1^2 \\
&\quad - r^3 s^3 (r + s)(r^2 + s^2) e_1 e_2 e_2^3 + r^6 s^6 e_2 e_1^4 = 0.
\end{aligned}$$

Once again, let us further define the following

$$\begin{aligned}
\Delta(e_1) &= e_1 \otimes 1 + k_1^2 k_2^{-1} \otimes e_1; \\
\Delta(e_2) &= e_2 \otimes 1 + k_1^{-3} k_2^2 \otimes e_2; \\
\Delta(k_1) &= k_1 \otimes k_1, & \Delta(k_2) &= k_2 \otimes k_2; \\
S(e_1) &= -k_1^2 k_2^{-1} e_1, & S(e_2) &= -k_1^{-3} k_2^2 e_2; \\
S(k_1) &= k_1^{-1}, & S(k_2) &= k_2^{-1}; \\
\epsilon(e_1) &= \epsilon(e_2) = 0, & \epsilon(k_1) &= \epsilon(k_2) = 1.
\end{aligned}$$

Then, it is easy to see that the above operators define a Hopf algebra structure on the augmented Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . Furthermore, the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$  has a PBW basis as described below.

**Proposition 1.2.** *The set*

$$\{X_1^a X_2^b X_3^c X_4^d X_5^e X_6^f k_1^m k_2^n | a, b, c, d, e, f \in \mathbb{Z}_{\geq 0}, m, n \in \mathbb{Z}\}$$

*forms a basis of  $\check{U}_{r,s}^{\geq 0}(G_2)$  over the base field  $\mathbb{C}$ .*

□

## 2. ALGEBRA AUTOMORPHISM GROUP OF THE HOPF ALGEBRA

$$\check{U}_{r,s}^{\geq 0}(G_2)$$

In this section, we will determine the algebra automorphism group of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . We will closely follow the approach used in [4]. Note that such an approach has been adopted to investigate the automorphism group of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(sl_3)$  in [7]. Similar work has also appeared in [8]. It is no surprise that we will derive very similar results to those obtained in [7, 8].

Let us denote by  $\theta$  an algebra automorphism of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . Since the elements  $k_1, k_2$  are invertible elements in the algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$  and  $\theta$  is an algebra automorphism of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ , the images  $\theta(k_1), \theta(k_2)$  of the invertible elements  $k_1, k_2$  are invertible elements in the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . Note that it is easy

to check that the only invertible elements of the algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$  are of the form  $\lambda k_1^m k_2^n$ ,  $\lambda \in \mathbb{C}^*$ ,  $m, n \in \mathbb{Z}$ . Therefore, the elements  $\theta(k_1)$  and  $\theta(k_2)$  can be expressed as follows

$$\theta(k_1) = \lambda_1 k_1^x k_2^y, \quad \theta(k_2) = \lambda_2 k_1^z k_2^w$$

for some  $\lambda_1, \lambda_2 \in \mathbb{C}^*$  and some  $x, y, z, w \in \mathbb{Z}$ .

Note that we can also associate an invertible  $2 \times 2$  matrix to the algebra automorphism  $\theta$ ; and we will denote this matrix by  $M_\theta = (M_{ij})$ . As a matter of fact, we will define this matrix by the entries as follows

$$M_{11} = x, M_{12} = y, M_{21} = z, M_{22} = w.$$

Due to the fact that the mapping  $\theta$  is an algebra automorphism, the determinant of the corresponding matrix  $M_\theta$  is either 1 or  $-1$ . That is, we shall have that

$$xw - yz = \pm 1.$$

In terms of the PBW basis of  $\check{U}_{r,s}^{\geq 0}(G_2)$ , we can further express the images of the generators  $e_1, e_2$  of  $\check{U}_{r,s}^{\geq 0}(G_2)$  under the algebra automorphism  $\theta$  as follows

$$\theta(e_l) = \sum_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3, \beta_l^4, \beta_l^5, \beta_l^6} \gamma_{m_l n_l \beta_l^1 \beta_l^2 \beta_l^3 \beta_l^4 \beta_l^5 \beta_l^6} k_1^{m_l} k_2^{n_l} X_1^{\beta_l^1} X_2^{\beta_l^2} X_3^{\beta_l^3} X_4^{\beta_l^4} X_5^{\beta_l^5} X_6^{\beta_l^6}$$

where  $\gamma_{m_l n_l \beta_l^1 \beta_l^2 \beta_l^3 \beta_l^4 \beta_l^5 \beta_l^6}$  are chosen from  $\mathbb{C}^*$ , and  $m_l, n_l$  are chosen from  $\mathbb{Z}$ , and  $\beta_l^1, \beta_l^2, \beta_l^3, \beta_l^4, \beta_l^5$  and  $\beta_l^6$  are chosen from  $\mathbb{Z}_{\geq 0}$ .

In the rest of this section, we prove that  $\theta$  is actually defined in a simple and specific way. First of all, we are going to establish some identities, whose proofs involve straightforward verifications; and we will not record these verifications.

**Lemma 2.1.** *For  $l = 1, 2$ , the following identities shall hold*

$$\begin{aligned} & k_1^x k_2^y X_1^{\beta_l^1} X_2^{\beta_l^2} X_3^{\beta_l^3} X_4^{\beta_l^4} X_5^{\beta_l^5} X_6^{\beta_l^6} \\ &= (r^{-1})^{x(-3\beta_l^1 - 2\beta_l^2 - 3\beta_l^3 - \beta_l^4 + \beta_l^6) + y(-6\beta_l^1 - 3\beta_l^2 - 3\beta_l^3 + 3\beta_l^5 + 3\beta_l^6)} \\ & \quad (s^{-2})^{x(-3\beta_l^1 - 2\beta_l^2 - 3\beta_l^3 - \beta_l^4 + \beta_l^6) + y(-6\beta_l^1 - 3\beta_l^2 - 3\beta_l^3 + 3\beta_l^5 + 3\beta_l^6)} \\ & \quad X_1^{\beta_l^1} X_2^{\beta_l^2} X_3^{\beta_l^3} X_4^{\beta_l^4} X_5^{\beta_l^5} X_6^{\beta_l^6} k_1^x k_2^y. \end{aligned}$$

□

Now we have the following proposition, which characterizes the nature of the matrix  $M_\theta$ .

**Proposition 2.1.** *Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(G_2))$  be an algebra automorphism of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ , then we have  $M_\theta \in GL(2, \mathbb{Z}_{\geq 0})$ .*

**Proof:** Since  $k_1 e_1 = r^{-1} s^{-2} e_1 k_1$ ,  $k_2 e_1 = r^{-1} s^{-1} e_1 k_2$  and  $\theta$  is an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(G_2)$ , we have the following

$$\begin{aligned}\theta(k_1)\theta(e_1) &= r^{-1} s^{-2} \theta(e_1) \theta(k_1); \\ \theta(k_2)\theta(e_1) &= r^{-3} s^{-3} \theta(e_1) \theta(k_2).\end{aligned}$$

Using the previous lemma, we shall have the following identities

$$\begin{aligned}x(-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6) + y(-6\beta_{11}^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6) &= 1; \\ x(-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6) + y(-6\beta_{11}^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6) &= 2; \\ x(-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6) + y(-6\beta_{11}^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6) &= -3; \\ x(-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6) + y(-6\beta_{11}^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6) &= -3; \\ z(-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6) + w(-6\beta_{11}^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6) &= 3; \\ z(-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6) + w(-6\beta_{11}^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6) &= 3; \\ z(-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6) + w(-6\beta_{11}^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6) &= -6; \\ z(-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6) + w(-6\beta_{11}^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6) &= -3.\end{aligned}$$

After some combinations and simplifications of these equations, we shall have the following system of equations:

$$\begin{aligned}x(\beta_1^2 + 3\beta_1^3 + 2\beta_1^4 + 3\beta_1^5 + \beta_1^6) + y(3\beta_1^1 + 3\beta_1^2 + 6\beta_1^3 + 3\beta_1^4 + 3\beta_1^5) &= 1; \\ x(\beta_2^2 + 3\beta_2^3 + 2\beta_2^4 + 3\beta_2^5 + \beta_2^6) + y(3\beta_2^1 + 3\beta_2^2 + 6\beta_2^3 + 3\beta_2^4 + 3\beta_2^5) &= 0; \\ z(\beta_1^2 + 3\beta_1^3 + 2\beta_1^4 + 3\beta_1^5 + \beta_1^6) + w(3\beta_1^1 + 3\beta_1^2 + 6\beta_1^3 + 3\beta_1^4 + 3\beta_1^5) &= 0; \\ z(\beta_2^2 + 3\beta_2^3 + 2\beta_2^4 + 3\beta_2^5 + \beta_2^6) + w(3\beta_2^1 + 3\beta_2^2 + 6\beta_2^3 + 3\beta_2^4 + 3\beta_2^5) &= 3.\end{aligned}$$

Now let us define a  $2 \times 2$ -matrix  $B = (b_{ij})$  by setting the entries of  $B$  as follows:

$$\begin{aligned} b_{11} &= \beta_1^2 + 3\beta_1^3 + 2\beta_1^4 + 3\beta_1^5 + \beta_1^6; \\ b_{21} &= 3\beta_1^1 + 3\beta_1^2 + 6\beta_1^3 + 3\beta_1^4 + 3\beta_1^5; \\ b_{12} &= \beta_1^2 + 3\beta_1^3 + 2\beta_1^4 + 3\beta_1^5 + \beta_1^6; \\ b_{22} &= 3\beta_2^1 + 3\beta_2^2 + 6\beta_2^3 + 3\beta_2^4 + 3\beta_2^5. \end{aligned}$$

Thus we shall have the following

$$M_\theta B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

which implies that we have

$$M_\theta^{-1} = \begin{pmatrix} b_{11} & b_{12}/3 \\ b_{21} & b_{22}/3 \end{pmatrix}.$$

Let us denote by  $M_{\theta^{-1}}$  the matrix associated to the inverse of  $\theta$ , then we have  $M_{\theta^{-1}} = M_\theta^{-1}$ . Since all the entries  $b_{11}, b_{12}, b_{21}, b_{22}$  of the matrix  $B$  are all nonnegative integers, we know that the matrix  $M_{\theta^{-1}}$  is indeed in the group  $GL(2, \mathbb{Z}_{\geq 0})$ . Apply this process to the algebra automorphism  $\theta^{-1}$ , we have that the matrix  $M_\theta$  is in  $GL(2, \mathbb{Z}_{\geq 0})$ . So we have proved the proposition.  $\square$

In addition, please note that the following important lemma was already established in the reference [4]. This lemma applies to our case as well.

**Lemma 2.2.** *If  $M$  is a matrix in  $GL(n, \mathbb{Z}_{\geq 0})$  such that its inverse matrix  $M^{-1}$  is also in  $GL(n, \mathbb{Z}_{\geq 0})$ , then we have  $M = (\delta_{i\sigma(j)})_{i,j}$ , where  $\sigma$  is an element of the symmetric group  $\mathbb{S}_n$ .*

$\square$

Based on **Proposition 2.1** and **Lemma 2.2**, it is easy to see that we have the following result.

**Corollary 2.1.** *Suppose that  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(G_2))$  is an algebra automorphism of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . Then for  $l = 1, 2$ , we have*

$$\theta(k_l) = \lambda_l k_{\sigma(l)}$$

where  $\sigma \in \mathbb{S}_2$  and  $\lambda_l \in \mathbb{C}^*$ .

$\square$

To proceed, we need some further preparations. Suppose that we have  $\theta(k_1) = \lambda_1 k_1$  and  $\theta(k_2) = \lambda_2 k_2$ . Then we have the following

**Lemma 2.3.** *The following identities hold*

$$\begin{aligned}
-3\beta_1^1 - 2\beta_1^2 - 3\beta_1^3 - \beta_1^4 + \beta_1^6 &= 1; \\
-3\beta_1^1 - \beta_1^2 + \beta_1^4 + 3\beta_1^5 + \beta_1^6 &= 2; \\
-6\beta_1^1 - 3\beta_1^2 - 3\beta_1^3 + 3\beta_1^5 + 3\beta_1^6 &= 3; \\
-3\beta_1^1 + 3\beta_1^3 + 3\beta_1^4 + 6\beta_1^5 + 3\beta_1^6 &= 3; \\
-3\beta_2^1 - 2\beta_2^2 - 3\beta_2^3 - \beta_2^4 + \beta_2^6 &= -3; \\
-3\beta_2^1 - \beta_2^2 + \beta_2^4 + 3\beta_2^5 + \beta_2^6 &= -3; \\
-6\beta_2^1 - 3\beta_2^2 - 3\beta_2^3 + 3\beta_2^5 + 3\beta_2^6 &= -6; \\
-3\beta_2^1 + 3\beta_2^3 + 3\beta_2^4 + 6\beta_2^5 + 3\beta_2^6 &= -3.
\end{aligned}$$

□

Moreover, the identities in the previous lemma imply the following

**Lemma 2.4.** *The following identities hold*

$$\begin{aligned}
\beta_1^2 + 3\beta_1^3 + 2\beta_1^4 + 3\beta_1^5 + \beta_1^6 &= 1; \\
3\beta_1^1 + 3\beta_1^2 + 6\beta_1^3 + 3\beta_1^5 &= 0; \\
\beta_2^2 + 3\beta_2^3 + 2\beta_2^4 + 3\beta_2^5 + \beta_2^6 &= 1; \\
3\beta_2^1 + 3\beta_2^2 + 6\beta_2^3 + 3\beta_2^5 &= 0.
\end{aligned}$$

*In particular, we have the following*

$$\begin{aligned}
\beta_1^1 = \beta_1^2 = \beta_1^3 = \beta_1^4 = \beta_1^5 = 0, \quad \beta_1^6 &= 1; \\
\beta_2^2 = \beta_2^3 = \beta_2^4 = \beta_2^5 = \beta_2^6 = 0, \quad \beta_2^1 &= 1.
\end{aligned}$$

□

Similarly, if we assume that we have  $\theta(k_1) = \lambda_1 k_2$  and  $\theta(k_2) = \lambda_2 k_1$ , then we shall have the following

**Lemma 2.5.**

$$\begin{aligned}
\beta_1^2 = \beta_1^3 = \beta_1^4 = \beta_1^5 = \beta_1^6 = 0, \quad \beta_1^1 &= 1; \\
\beta_2^1 = \beta_2^2 = \beta_2^3 = \beta_2^4 = \beta_2^5 = 0, \quad \beta_2^6 &= 1.
\end{aligned}$$

□

Follows from the previous two lemmas, we can easily have the following result

**Proposition 2.2.** *Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(G_2))$  be an algebra automorphism of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . Then for  $l = 1, 2$ , we have*

$$\theta(e_l) = \gamma_l k_1^{m_l} k_2^{n_l} e_{\sigma(l)}$$

where  $\gamma_l \in \mathbb{C}^*$  and  $m_l, n_l \in \mathbb{Z}$ .



□

The following result will further demonstrate that the two generators  $e_1, e_2$  of  $\check{U}_{r,s}^{\geq 0}(G_2)$  can not be exchanged by any algebra automorphism  $\theta$  of  $\check{U}_{r,s}^{\geq 0}(G_2)$ . In particular, we have the following result

**Corollary 2.2.** *Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(G_2))$  be an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(G_2)$ . Then for  $l = 1, 2$ , we have the following*

$$\theta(k_l) = \lambda_l k_l, \theta(e_l) = \gamma_l k_1^{m_l} k_2^{n_l} e_l$$

where  $\lambda_l, \gamma_l \in \mathbb{C}^*$  and  $m_l, n_l \in \mathbb{Z}$ .

**Proof:** Suppose that  $\theta(k_1) = \lambda_1 k_2$  and  $\theta(e_2) = \gamma_1 k_1^{m_1} k_2^{n_1} e_2$ . Since we have  $\theta(k_1)\theta(e_1) = r^{-1}s^{-2}\theta(e_1)\theta(k_1)$ , we have the following

$$\lambda_1 k_2 \gamma_1 k_1^{m_1} k_2^{n_1} e_2 = r^{-1}s^{-2} \gamma_1 k_1^{m_1} k_2^{n_1} e_2 \lambda_1 k_2.$$

Note that  $k_2 e_2 = r^6 s^3 e_2 k_2$ , then we got a contradiction. Therefore, we have proved the statement as desired. □

Now we will further establish some identities via direct calculations and we will skip the detailed calculations here.

**Lemma 2.6.** *We have the following identities:*

$$\begin{aligned} (k_1^a k_2^b e_1)^4 (k_1^c k_2^d e_2) &= r^{6a+18b+4c+12d} s^{12a+18b+8c+12d} k_1^{a+c} k_2^{b+d} e_1^4 e_2; \\ (k_1^a k_2^b e_1)^3 (k_1^c k_2^d e_2) (k_1^a k_2^b e_1) &= r^{3a+12b+3c+9d} s^{9a+15b+6c+9d} k_1^{4a+c} k_2^{4b+d} e_1^3 e_2 e_1; \\ (k_1^a k_2^b e_1)^2 (k_1^c k_2^d e_2) (k_1^a k_2^b e_1)^2 &= r^{6b+2c+6d} s^{6a+12b+4c+6d} k_1^{4a+c} k_2^{4b+d} e_1^2 e_2 e_1^2; \\ (k_1^a k_2^b e_1) (k_1^c k_2^d e_2) (k_1^a k_2^b e_1)^3 &= r^{-3a+c+3d} s^{3a+9b+2c+3d} k_1^{4a+c} k_2^{4b+d} e_1 e_2 e_1^3; \\ (k_1^c k_2^d e_2) (k_1^a k_2^b e_1)^4 &= r^{-6a-6b} s^{6b} k_1^{4a+c} k_2^{4b+d} e_2 e_1^4. \end{aligned}$$

□

Similarly, we can also have the following lemma, whose proof will be skipped.

**Lemma 2.7.** *The following identities hold.*

$$\begin{aligned} (k_1^c k_2^d e_2)^2 (k_1^a k_2^b e_1) &= r^{-6a-12b-3c-6d} s^{-6a-6b-3c-3d} k_1^{a+2c} k_2^{b+2d} e_2^2 e_1; \\ (k_1^c k_2^d e_2) (k_1^a k_2^b e_1) (k_1^c k_2^d e_2) &= r^{-3a-6b-2c-3d} s^{-3a-3b-c} k_1^{a+2c} k_2^{b+2d} e_2 e_1 e_2; \\ (k_1^a k_2^b e_1)^2 (k_1^c k_2^d e_2)^2 &= r^{-c+6d} s^{c+3d} k_1^{a+2c} k_2^{b+2d} e_1 e_2^2. \end{aligned}$$

□

Now we are ready to prove one of the main results of this paper, which describes the group of algebra automorphisms of the algebra  $\text{Hopf } \check{U}_{r,s}^{\geq 0}(G_2)$ . Namely, we have the following

**Theorem 2.1.** *Let  $\theta \in \text{Aut}_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(G_2))$  be an algebra automorphism of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . Then for  $l = 1, 2$ , we have the following*

$$\theta(k_l) = \lambda_l k_l, \quad \theta(e_1) = \gamma_1 k_1^a K_2^b e_1, \quad \theta(e_2) = \gamma_2 k_1^c k_2^d e_2$$

where  $\lambda_l, \gamma_l \in \mathbb{C}^*$  and  $a, b, c, d \in \mathbb{Z}$  such that  $c = 3b, a + 3b + d = 0$ .

**Proof:** Let  $\theta$  be an algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(G_2)$  and suppose that

$$\theta(e_1) = \gamma_1 k_1^a k_2^b e_1, \quad \theta(e_2) = \gamma_2 k_1^c k_2^d e_2.$$

Note that the generators  $e_1, e_2$  satisfy the following two-parameter quantum Serre relations

$$\begin{aligned} e_2^2 e_1 - (r^{-3} + s^{-3}) e_2 e_1 e_2 + r^{-32} s^{-3} e_1 e_2^2 &= 0; \\ e_1^4 e_2 - (r + s)(r^2 + s^2) e_1^3 e_2 e_1 + rs(r^2 + rs + s^2) e_1^2 e_2 e_1^2 \\ - r^3 s^3 (r + s)(r^2 + s^2) e_1 e_2 e_2^3 + r^6 s^6 e_2 e_1^4 &= 0. \end{aligned}$$

Since  $\theta$  is an algebra automorphism of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ , we know that  $\theta$  preserves the quantum Serre relations. In particular, we can derive the following system of equations via using the previous lemmas.

$$\begin{aligned} 6a + 18b + 4c + 12d &= 3a + 12b + 3c + 9d; \\ 6b + 2c + 6d &= 3a + 12b + 3c + 9d; \\ -3a + c + 3d &= 3a + 12b + 3c + 9d; \\ -6a - 6b &= 3a + 12b + 3c + 9d; \\ 12a + 18b + 8c + 12d &= 9a + 15b + 6c + 9d; \\ 6a + 12b + 4c + 6d &= 9a + 15b + 6c + 9d; \\ 3a + 9b + 2c + 3d &= 9a + 15b + 6c + 9d; \\ 6b &= 9a + 15b + 6c + 9d; \\ -6a - 12b - 3c - 6d &= -3a - 6b - 2c - 3d; \\ -c &= -3a - 6b - 2c - 3d; \\ -6a - 6b - 3c - 3d &= -3a - 3b - c; \\ c + 3d &= -3a - 3b - c. \end{aligned}$$

Solving the previous system of equations, we shall obtain the following system of equations

$$\begin{aligned} 3b &= c; \\ a + c + d &= 0. \end{aligned}$$

Therefore, we have proved the theorem as desired.  $\square$

3. HOPF ALGEBRA AUTOMORPHISMS OF  $\check{U}_{r,s}^{\geq 0}(G_2)$ 

In this section, we will determine all the Hopf algebra automorphisms of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(G_2)$ . We denote by  $Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(G_2))$  the group of all Hopf algebra automorphisms of  $\check{U}_{r,s}^{\geq 0}(G_2)$ . In particular, we shall prove that  $Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(G_2))$  is isomorphic to a torus of rank 2.

To finish the task of this section, we need to establish the following result

**Theorem 3.1.** *Let  $\theta \in Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(G_2))$ . Then for  $l = 1, 2$ , we have the following*

$$\theta(k_l) = k_l, \quad \theta(e_l) = \gamma_l e_l,$$

for some  $\gamma_l \in \mathbb{C}^*$ . In particular, we have

$$Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(G_2)) \cong (\mathbb{C}^*)^2.$$

**Proof:** Let  $\theta \in Aut_{Hopf}(\check{U}_{r,s}^{\geq 0}(G_2))$  denote a Hopf algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(G_2)$ , then we have  $\theta \in Aut_{\mathbb{C}}(\check{U}_{r,s}^{\geq 0}(G_2))$ . According to **Theorem 2.1.**, we shall have the following

$$\begin{aligned} \theta(k_l) &= \lambda_l k_l; \\ \theta(e_1) &= \gamma_1 k_1^a k_2^b e_1; \\ \theta(E_2) &= \gamma_2 k_1^c k_2^d e_2; \end{aligned}$$

for some  $\lambda_l, \gamma_l \in \mathbb{C}^*$  for  $l = 1, 2$ , and  $a, b, c, d \in \mathbb{Z}$  such that  $3b = c, a + c + d = 0$ .

First of all, we need to show that we have  $\lambda_l = 1$  for  $l = 1, 2$ . Since  $\theta$  is a Hopf algebra automorphism, we shall have the following

$$(\theta \otimes \theta)(\Delta(k_l)) = \Delta(\theta(k_l))$$

for  $l = 1, 2$ , which implies the following

$$\lambda_l^2 = \lambda_l$$

for  $l = 1, 2$ . Thus, we have  $\lambda_l = 1$  for  $l = 1, 2$  as desired.

Second of all, we need to show that we have  $a = b = c = d = 0$ . Note that we have the following

$$\begin{aligned} \Delta(\theta(e_1)) &= \Delta(\gamma_1 k_1^a k_2^b e_1) \\ &= \Delta(\gamma_1 k_1^a k_2^b) \Delta(e_1) \\ &= \gamma_1 (k_1^a k_2^b \otimes k_1^a k_2^b) (e_1 \otimes 1 + k_1^2 k_2^{-1} \otimes e_1) \\ &= \gamma_1 k_1^a k_2^b e_1 \otimes k_1^a k_2^b + \gamma_1 k_1^a k_2^b k_1^2 k_2^{-1} \otimes k_1^a k_2^b e_1 \\ &= \theta(e_1) \otimes k_1^a k_2^b + k_1^a k_2^b k_1^2 k_2^{-1} \otimes \theta(e_1). \end{aligned}$$

In addition, we also have the following

$$\begin{aligned}
(\theta \otimes \theta)(\Delta(e_1)) &= (\theta \otimes \theta)(e_1 \otimes 1 + k_1^2 k_2^{-1} \otimes e_1) \\
&= \theta(e_1) \otimes 1 + \theta(k_1^2 k_2^{-1}) \otimes \theta(e_1) \\
&= \theta(e_1) \otimes 1 + k_1^2 k_2^{-1} \otimes \theta(e_1).
\end{aligned}$$

Since we have  $\Delta(\theta(e_1)) = (\theta \otimes \theta)\Delta(e_1)$ , we shall have  $a = b = 0$ . Note that we have  $3b = c$  and  $a + c + d = 0$ , thus we have  $a = b = c = d = 0$  as desired.

Conversely, it is obvious to see that the algebra automorphism  $\theta$  defined by  $\theta(k_l) = k_l$  and  $\theta(e_l) = \gamma_l e_l$  for  $l = 1, 2$  is a Hopf algebra automorphism of  $\check{U}_{r,s}^{\geq 0}(G_2)$ . So we have proved the theorem.  $\square$

## REFERENCES

- [1] Bergeron, N., Gao, Y. and Hu, N., Drinfeld doubles and Lusztig's symmetries of two-parameter quantum groups, *J. Algebra* **301** (2006), no. 1, 378–405.
- [2] Benkart, G. and Witherspoon, S., Two-parameter quantum groups and Drinfeld doubles, *Algebr. Represent. Theory*, **7** (2004), 261–286.
- [3] Benkart, G. and Witherspoon, S., Representations of two-parameter quantum groups and Schur–Weyl duality, Hopf algebras, Lecture Notes in Pure and Appl. Math., **237**, pp. 65–92, Dekker, New York, 2004.
- [4] Fleury, O., Automorphisms of  $\check{U}_q(\mathfrak{b}^+)$ , *Beiträge Algebra and Geom.*, **Vol 38(2)** (1997), 343–356.
- [5] Hu, N.H., Shi, Q., The two-parameter quantum group of exceptional type  $G_2$  and Lusztig's symmetries, *Pacific J. Math.*, **230** (2007), no. 2, 327–345.
- [6] Hu, N.H., Wang, X.L., Convex PBW-type Lyndon basis and restricted two-parameter quantum groups of type  $G_2$ , *Pacific J. Math.*, **241** (2009), no. 2, 243–273.
- [7] Tang, X., (Hopf) algebra automorphisms of the Hopf algebra  $\check{U}_{r,s}^{\geq 0}(sl_3)$ , submitted.
- [8] Tang, X., Derivations of the two-parameter quantized enveloping algebra  $U_{r,s}^+(B_2)$ , preprint.

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